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# Superalgebra and Harmonic Oscillator with Constant Positive Curvature

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**Abstract** We study the Schrödinger equation for the isotropic oscillator in three-dimensional space with constant positive curvature. So, the description of solutions for the corresponding Schrödinger equation based on the spherical coordinates. By comparing the Schrödinger equation of harmonic oscillator in constant positive curvature to the associated Jacobi polynomial, we obtain the energy spectrum and wave function. The associated Jacobi equation help us to factorize the Schrödinger equation for the isotropic oscillator. The first order equation from factorization method lead us to define the raising and lowering operators. These operators are supersymmetric structure related to the Hamiltonian partner, thus we obtain the corresponding supercharges.

**Keywords** Superalgebra · Harmonic oscillator · Constant curvature · Raising and lowering operators

## 1 Introduction

The accidental degeneracy in the space of constant curvature for the first time discussed by Schrödinger [1, 2], Infeld [3] and Stivenson [4]. This subject has interested many researcher. Because it connect to nontrivial realization of hidden symmetry and also apply to construct the many-particle wave functions [5], nonrelativistic models of quark systems [6, 7] and solutions of two-center problem [8]. Also the accidental degeneracy in this system are discussed by Refs. [9–15]. They have shown that the complete degeneracy of spectrum of the

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Coulomb problem and harmonic oscillator on the three dimensional sphere in the orbital and azimuthal quantum number is caused by an additional integral of motion. However, in contrast with the flat the integrals of motion for the coulomb problem and isotropic oscillator do not simple form the algebra as the relevant commutators. The group of hidden symmetry of this systems with accidental degeneracy give us motivation to discuss the isotropic oscillator with constant positive curvature. Also note that the concept of shape invariance has extended to ordinary differential equations. In that case the second order differential operator will decompose the multiplication of raising and lowering operators [16–21]. In this paper, we use the factorization method and shape invariance of the associated Jacobi equation with respect to parameters $n$ and $m$ and obtain the factorized Schrödinger equations for the harmonic oscillator in constant positive curvature. Also we obtain the shape invariance relation for the corresponding potential. So, the paper organize as follows: Sect. 2 presents the general form of the isotropic oscillator in spherical coordinates with spaces of constant curvature. Section 3 we use the associated Jacobi equation and solve the corresponding equation. Section 4 by using the factorization method from associated Jacobi equation we obtain the raising and lowering operators, and finally we obtain the superalgebra which is important in supersymmetry system.

## 2 Isotropic Oscillator on the Constant Curvature

As we know the three-dimensional space of constant positive curvature can also be realized geometrically on the three-dimensional sphere $S_3$ of the radius $R$, embedded into the four-dimensional Euclidean space,

$$\xi_0{}^2 + \xi_i \xi_i = R^2. \tag{1}$$

We note that the relation between the coordinates $x_i$ in the tangent space and $\xi_\mu$ ($\mu = 0, 1, 2, 3$) is given by,

$$\xi_i = \frac{x_i}{\sqrt{1 + \frac{r^2}{R^2}}}, \qquad \xi_0 = \frac{R}{\sqrt{1 + \frac{r^2}{R^2}}}, \tag{2}$$

where the coordinates $\xi_i$ change in the region $\xi_i \xi_i \leq R^2$.

Now we are going to write the general form of isotropic oscillator potential in space of constant curvature. By using the $r^2 = x_1{}^2 + x_2{}^2 + x_3{}^2$ and (2) the following potential,

$$V(r) = \frac{\mu\omega^2}{2} r^2, \tag{3}$$

change to,

$$V(r) = V(\xi) = \frac{1}{2}\mu\omega^2 \frac{\xi^2}{1 - \frac{\xi^2}{R^2}}. \tag{4}$$

In the spherical system of coordinates we have,

$$\begin{aligned} &\xi_1 = R\sin\psi\sin\theta\cos\phi, \qquad \xi_2 = R\sin\psi\sin\theta\sin\phi, \\ &\xi_3 = R\sin\psi\cos\theta, \qquad \xi_0 = R\cos\psi, \end{aligned} \tag{5}$$

where $0 \leq \psi < \pi$, $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$.

So, the oscillator potential (4) in spherical system is,

$$V(\psi)=\frac{\mu\omega^2R^2}{2}\tan^2\psi, \tag{6}$$

which is very similar to the nonlinear harmonic oscillator on the flat space.

## 3 Solution of the Schrödinger Equation

In order to solve the Schrödinger equation, we need to write the corresponding equation with potential (6) on constant curvature as follow,

$$\left[-\frac{\hbar^2}{2\mu}\Delta+V\right]\Psi=E\Psi, \tag{7}$$

where $\Delta$ is the Laplace–Beltrami operator and given by,

$$\Delta=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\sqrt{g}g^{ik}\frac{\partial}{\partial x^k} \tag{8}$$

so the metric is,

$$ds^2=g_{ik}dx^idx^k \tag{9}$$

where $g=\det(g_{ik})$, $g^{ik}=(g_{ik})^{-1}$ and $r^2=x_ix_i$ $(i,k=1,2,3)$. By using (5), (6), (7) and (8) one can obtain the Schrödinger equation (7) as the following,

$$\left[\frac{1}{\sin^2\psi}\frac{\partial}{\partial\psi}\sin^2\psi\frac{\partial}{\partial\psi}\right]\Psi+\frac{2\mu R^2}{\hbar^2}\left[E-\frac{\hbar^2}{2\mu R^2}\frac{m(m+1)}{\sin^2\psi}-\frac{\mu\omega^2R^2}{2}\tan^2\psi\right]\Psi=0. \tag{10}$$

For simplicity we take following variable,

$$\left(m+\frac{1}{2}\right)^2={l_1}^2,\qquad \frac{\mu^2\omega^2R^4}{\hbar^2}+\frac{1}{4}={l_2}^2,\qquad \frac{2\mu R^2E}{\hbar^2}+\frac{\mu^2\omega^2R^4}{\hbar^2}+1=\epsilon, \tag{11}$$

and

$$\Psi(\psi)=\frac{\Phi(\psi)}{\sin\psi}. \tag{12}$$

We put (11) and (12) to (10) and finally we will arrive at,

$$\frac{d^2\Phi}{d\psi^2}+\left[\epsilon-\frac{({l_1}^2-\frac{1}{4})}{\sin^2\psi}-\frac{({l_2}^2-\frac{1}{4})}{\cos^2\psi}\right]\Phi=0. \tag{13}$$

Now we are going to choice the variable $x=\cos 2\psi$ and $\Phi=F(x)P_{n,m}^{(\alpha,\beta)}(x)$ and compare to the following associated Jacobi equation,

$$(1-x^2)P''_{n,m}(x)-[\alpha-\beta+(\alpha+\beta+2)x]P'_{n,m}$$
$$+\left[n(\alpha+\beta+n+1)-\frac{m(\alpha+\beta+m+(\alpha-\beta)x)}{1-x^2}\right]P_{n,m}(x)=0, \tag{14}$$

so, we obtain the energy spectrum and wavefunction as follow,

$$E=\frac{\hbar^2}{2\mu R^2}\left[2\left(2n-m+\frac{3}{2}\right)l_2+(2n-m)^2+3(2n-m)+\frac{3}{2}\right], \tag{15}$$

and

$$\Psi(\psi)=a_{n,m}\frac{1}{\sin\psi}(1+\cos 2\psi)^{\frac{2\beta+1}{4}}(1-\cos 2\psi)^{\frac{2\alpha+1}{4}}P_{n,m}^{(\alpha,\beta)}(\cos 2\psi) \tag{16}$$

where $\alpha=l_1-m$, $\beta=l_2-m$ and $a_{n,m}$ normalization factor.

Here we note that by using $N\equiv 2n-m$ and $\nu\equiv l_2-\frac{1}{2}$ in (15), we have,

$$E_N=\frac{\hbar^2}{2\mu}\left[\frac{(N+1)(N+3)}{R^2}+\frac{2\nu}{R^2}\left(N+\frac{3}{2}\right)\right]. \tag{17}$$

In flat space $R\to\infty$ the energy spectrum will be as,

$$E=\hbar\omega\left(N+\frac{3}{2}\right), \tag{18}$$

which is agree with three dimensional harmonic oscillator.

And the associated Jacobi functions $P_{n,m}^{(\alpha,\beta)}(x)$ as the solution of the differential equation have the following Rodrigues representation,

$$P_{n,m}^{(\alpha,\beta)}(x)=\frac{a_{n,m}(\alpha,\beta)}{(1-x)^{\alpha+\frac{m}{2}}(1+x)^{\beta+\frac{m}{2}}}\left(\frac{d}{dx}\right)^{n-m}((1-x)^{\alpha+n}(1+x)^{\beta+n}). \tag{19}$$

## 4 Supersymmetry Algebra and Supercharges

In order to obtain the generators of supersymmetry algebra, we have to calculate the raising and lowering operators corresponding to harmonic oscillator with constant positive curvature. As mentioned in Refs. [16–21] and (14) we can write the (10) as the following,

$$\begin{aligned}&A_{n,m}^{+}(\psi)A_{n,m}^{-}(\psi)P_{n,m}(\psi)=B_{n,m}P_{n,m}(\psi),\\&A_{n,m}^{-}(\psi)A_{n,m}^{+}(\psi)P_{n-1,m}(\psi)=B_{n,m}P_{n-1,m}(\psi),\end{aligned} \tag{20}$$

where

$$\begin{aligned}B_{n,m}&=\frac{4(n-m)(\frac{1}{2}+n)(l_2-m+n)(\frac{1}{2}+l_2+n)}{(\frac{1}{2}+l_2-m+2n)^2},\\A_{n,m}^{+}(\psi)&=\frac{-\sin 2\psi}{2}\frac{d}{d\psi}-\left(\frac{1}{2}+l_2-m+n\right)\cos 2\psi\\&\quad-\frac{(\frac{1}{2}-l_2+m)(\frac{1}{2}+l_2-n)}{\frac{1}{2}+l_2-m+2n},\\A_{n,m}^{-}(\psi)&=\frac{\sin 2\psi}{2}\frac{d}{d\psi}-n\cos 2\psi+\frac{(\frac{1}{2}-l_2+m)(n-m)}{\frac{1}{2}+l_2-m+2n}.\end{aligned} \tag{21}$$

By considering the shape invariance formalism with respect to parameter $m$, we have,

$$\begin{aligned} A_m^+(\psi)A_m^-(\psi)P_{n,m}(\psi) &= C_{n,m}P_{n,m}(\psi), \\ A_m^-(\psi)A_m^+(\psi)P_{n,m-1}(\psi) &= C_{n,m}P_{n,m-1}(\psi), \end{aligned} \tag{22}$$

where

$$C(n,m) = (n-m+1)\left(\frac{1}{2} + l_2 + n\right) \tag{23}$$

and $0 \le \psi \le \frac{\pi}{2}$, so we have

$$\begin{aligned} A_m^+ &= -\frac{1}{2}\frac{d}{d\psi} + (m+1)\cot 2\psi, \\ A_m^- &= \frac{1}{2}\frac{d}{d\psi} + \frac{[(m+\frac{1}{2}-l_2) + (\frac{1}{2}+l_2)\cos 2\psi]}{\sin 2\psi}. \end{aligned} \tag{24}$$

Therefore, we have obtained different types of the laddering relations for the harmonic oscillator with constant positive curvature. These raising and lowering operators help us to obtain the supercharges operators and also construct a representation of supersymmetry algebra. In here we consider two cases, first we discuss the raising and lowering operators which are factorized to $n$ and $m$ indices. In that case we rewrite the operators for the special case $m = l_2 + \frac{1}{2}$ as,

$$\begin{aligned} A_{n,m}^+(\psi) &= -\frac{\sin 2\psi}{2}\frac{d}{d\psi} + W_1, \\ A_{n,m}^-(\psi) &= \frac{\sin 2\psi}{2}\frac{d}{d\psi} + W_1, \end{aligned} \tag{25}$$

where superpotential is,

$$W_1 = -n\cos 2\psi + \frac{(n - l_2 - \frac{1}{2})}{2n}. \tag{26}$$

In order to obtain the supercharges we have to make the bosonic and fermionic partner Hamiltonian $H_1$ and $H_2$,

$$\begin{aligned} H_1 &= A_{n,m}^+ A_{n,m}^-, \\ H_2 &= A_{n,m}^- A_{n,m}^+, \end{aligned} \tag{27}$$

and

$$H_1 = \frac{P^2}{2\mu} + V_1, \qquad H_2 = \frac{P^2}{2\mu} + V_2, \tag{28}$$

where $V_1$ and $V_2$ are,

$$\begin{aligned} V_1 &= {W_1}^2 + \frac{\sin 2\psi}{2}{W_1}' = \left(-n\cos 2\psi + \frac{n - l_2 - \frac{1}{2}}{2n}\right)^2 + n\sin^2 2\psi, \\ V_2 &= {W_1}^2 - \frac{\sin 2\psi}{2}{W_1}' = \left(-n\cos 2\psi + \frac{n - l_2 - \frac{1}{2}}{2n}\right)^2 - n\sin^2 2\psi. \end{aligned} \tag{29}$$

In that case we have shape invariance if we change in $V_2$, $n \to -n$ and $l_2 \to l_2 - 2n$, so we can write

$$V_1(n, l_2) - V_2(-n, l_2 - 2n) = 0.$$

Now we are going to obtain the supercharges,

$$Q = \begin{pmatrix} 0 & 0 \\ A^{-}_{n,m} & 0 \end{pmatrix}, \qquad Q^{+} = \begin{pmatrix} 0 & A^{+}_{n,m} \\ 0 & 0 \end{pmatrix}, \tag{30}$$

where

$$A^{-}_{n,m} = \frac{\sin 2\psi}{2}\frac{d}{d\psi} - n\cos 2\psi + \left(\frac{n - l_2 - \frac{1}{2}}{2n}\right),$$

and

$$A^{+}_{n,m} = \frac{-\sin 2\psi}{2}\frac{d}{d\psi} - n\cos 2\psi + \left(\frac{n - l_2 - \frac{1}{2}}{2n}\right).$$

Now we discuss the second case which is factorized with respect to $m$, that case we rewrite the operators for the special case $m = l_2 - \frac{1}{2}$ as,

$$\begin{aligned} A^{+}_{m}(\psi) &= -\frac{1}{2}\frac{d}{d\psi} + W_1, \\ A^{-}_{m}(\psi) &= \frac{1}{2}\frac{d}{d\psi} + W_1, \end{aligned} \tag{31}$$

where superpotential is,

$$W_1 = \left(l_2 + \frac{1}{2}\right)\cot 2\psi. \tag{32}$$

The bosonic and fermionic partner Hamiltonian $H_1$ and $H_2$ will be as,

$$\begin{aligned} H_1 &= A^{+}_{m}A^{-}_{m}, \\ H_2 &= A^{-}_{m}A^{+}_{m}, \end{aligned} \tag{33}$$

and

$$H_1 = \frac{P^2}{2\mu} + V_1, \qquad H_2 = \frac{P^2}{2\mu} + V_2, \tag{34}$$

where $V_1$ and $V_2$ are,

$$\begin{aligned} V_1 &= W_1^2 + \frac{\sin 2\psi}{2}W_1{}' = \left[\left(l_2 + \frac{1}{2}\right)\cot 2\psi\right]^2 - \frac{(l_2 + \frac{1}{2})}{\sin 2\psi}, \\ V_2 &= W_1^2 - \frac{\sin 2\psi}{2}W_1{}' = \left[\left(l_2 + \frac{1}{2}\right)\cot 2\psi\right]^2 + \frac{(l_2 + \frac{1}{2})}{\sin 2\psi}. \end{aligned} \tag{35}$$

In that case, we have shape invariance if we change in $V_2$, $l_2 \to -l_2 - 1$, also we can write,

$$V_1(l_2) - V_2(l_2 - 1) = 0.$$

The corresponding supercharges as,

$$Q = \begin{pmatrix} 0 & 0 \\ A_m^- & 0 \end{pmatrix}, \qquad Q^+ = \begin{pmatrix} 0 & A_m^+ \\ 0 & 0 \end{pmatrix}, \tag{36}$$

where

$$A_m^- = \frac{1}{2}\frac{d}{d\psi} + \left(l_2 + \frac{1}{2}\right)\cot 2\psi$$

and

$$A_m^+ = -\frac{1}{2}\frac{d}{d\psi} + \left(l_2 + \frac{1}{2}\right)\cot 2\psi.$$

## 5 Conclusion

In this paper we discuss isotropic oscillator in constant curvature. We have obtained the energy spectrum and wavefunction. Using the factorization method we derive some raising and lowering operators. These lead us to introduce some supercharge operators. With the help of shape invariance condition it may be interesting to discuss the representation of superalgebra for oscillator in constant curvature.

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